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ON GENERALIZED TWO-DIMENSIONAL PLATE THEORY—II*

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Abstract—In elaboration of a recently proposed procedure for the derivation of two-dimensional shell theory from three-dimensional elasticity theory the special case of a flat plate is considered explicitly. The starting point of the work is a suitable version of elasticity theory including moment as well as force stresses. It is shown that a certain straightforward but not previously considered reduction of three-dimensional equilibrium and compatibility equations leads to suitable two-dimensional equilibrium and compatibility equations, while leaving the three-dimensional aspects of the problem in the form of a system of integro-differential constitutive equations. The derivation of two-dimensional constitutive equations then is one involving parametric expansion or iteration, in conjunction with the stipulation of smallest characteristic length large compared to plate thickness.

1. INTRODUCTION

The principal purpose of this paper is to describe, by means of a somewhat simpler special case of independent interest, a recent new approach to the problem of deriving two-dimensional shell theory from three-dimensional elasticity theory [5]. In applying this approach here to the special case of the initially uncurved shell, the two-dimensional results are developed appreciably further than was done for the general case in the earlier paper [5].

The main point of our recent approach to deriving two-dimensional shell theory is that by basing this derivation on a three-dimensional theory of elasticity which includes the consideration of moment stresses the task becomes simpler than on the basis of a theory without moment stresses. Mathematically, this simplification may be ascribed to the fact that in the theory including moment stresses all equilibrium and compatibility equations are first order differential equations, while in the theory without moment stresses one has to deal with a system including zeroth order, first order and second order differential equations. A physical reasoning, which was in fact the starting point of our earlier considerations, is that there ought to be advantages to deriving a two-dimensional theory in which forces and moments play an equally important role from a three-dimensional theory for which this is the case.

2. THE THREE-DIMENSIONAL PROBLEM

With reference to cartesian coordinates x, y, z we consider the space bounded by planes $z = \pm c$. The differential equations of the problem are six equilibrium equations

$$\begin{aligned}
&\sigma_{xx,x}+\sigma_{yx,y}+\sigma_{zx,z} = 0, \qquad \tau_{xx,x}+\tau_{yx,y}+\tau_{zx,z}+\sigma_{zx}-\sigma_{xz} = 0\\
&\sigma_{xy,x}+\sigma_{yy,y}+\sigma_{zy,z} = 0, \qquad \tau_{xy,x}+\tau_{yy,y}+\tau_{zy,z}+\sigma_{zy}-\sigma_{yz} = 0\\
&\sigma_{xz,x}+\sigma_{yz,y}+\sigma_{zz,z} = 0, \qquad \tau_{xz,x}+\tau_{yz,y}+\tau_{zz,z}+\sigma_{xy}-\sigma_{yx} = 0
\end{aligned} \tag{1}$$

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for nine force stress components σ and nine moment stress components τ, where for simplicity's sake volume forces and moments are assumed to be absent, together with eighteen compatibility equations which are written in the semi-integrated form (which follows from corresponding vectorial formulas in [5] or on the basis of explicit strain displacement relations in [6])

$$\begin{aligned} k_{xx} &= \kappa_{xx} + \int_0^z k_{zx,x}\,d\zeta, & k_{xy} &= \kappa_{xy} + \int_0^z k_{zy,x}\,d\zeta, \\ k_{yx} &= \kappa_{yx} + \int_0^z \kappa_{zx,y}\,d\zeta, & k_{yy} &= \kappa_{yy} + \int_0^z k_{zy,y}\,d\zeta, \\ k_{xz} &= \lambda_x + \int^z k_{zz,x}\,d\zeta, & k_{yz} &= \lambda_y + \int_0^z k_{zz,y}\,d\zeta, \end{aligned} \tag{2}$$

$$\kappa_{yx,x} - \kappa_{xx,y} = 0, \qquad \kappa_{yy,x} - \kappa_{xy,y} = 0, \qquad \lambda_{y,x} - \lambda_{x,y} = 0, \tag{3}$$

$$\begin{aligned} e_{xx} &= \varepsilon_{xx} + z\kappa_{xx} + \int_0^z [e_{zx} - (\zeta - z)k_{zx}]_{,x}\,d\zeta, \\ e_{xy} &= \varepsilon_{xy} + z\kappa_{xy} - \int_0^z k_{zz}\,d\zeta + \int_0^z [e_{zy} - (\zeta - z)k_{zy}]_{,x}\,d\zeta, \\ e_{yx} &= \varepsilon_{yx} + z\kappa_{yx} + \int_0^z k_{zz}\,d\zeta + \int_0^z [e_{zx} - (\zeta - z)k_{zx}]_{,y}\,d\zeta, \\ e_{yy} &= \varepsilon_{yy} + z\kappa_{yy} + \int_0^z [e_{zy} - (\zeta - z)k_{zy}]_{,y}\,d\zeta, \end{aligned} \tag{4}$$

$$e_{xz} = \gamma_x + \int_0^z k_{zx} d\zeta + \int_0^z e_{zz,x}\,d\zeta, \quad e_{yz} = \gamma_y + \int_0^z k_{zy}\,d\zeta + \int_0^z e_{zz,y}\,d\zeta,$$

and

$$\varepsilon_{yx,x} - \varepsilon_{xx,y} = \lambda_x, \qquad \varepsilon_{yy,x} - \varepsilon_{xy,y} = \lambda_y, \qquad \gamma_{y,x} - \gamma_{x,y} = \kappa_{xy} - \kappa_{yx}. \tag{5}$$

In this the e, ε and γ may be designated as force strains and the k, κ and λ as moment strains. The e and k are functions of x, y, z while the ε, γ, κ, λ depend on x and y only.

Equations (1), (2) and (4) are a system of eighteen equations for thirty-six functions σ, τ, e, k. A system of thirty-six equations for thirty-six unknowns is obtained upon complementing (1), (2) and (4) by eighteen stress–strain relations:

$$\sigma_{xx} = \frac{\partial A}{\partial e_{xx}}, \qquad \sigma_{xy} = \frac{\partial A}{\partial e_{xy}}, \ldots, \qquad \tau_{yz} = \frac{\partial A}{\partial k_{yz}}, \qquad \tau_{zz} = \frac{\partial A}{\partial k_{zz}}, \tag{6}$$

where A is a given function of the eighteen arguments e and k.

Equations (1) to (6) are further complemented by six boundary conditions for each of the two faces of the shell, of the form

$$z = \pm c, \qquad \sigma_{zx} = p_x^{\pm}, \qquad \sigma_{zy} = p_y^{\pm}, \ldots, \qquad \tau_{zz} = q_z^{\pm}, \tag{7}$$

where p and q are given functions of x and y.

Writing, on the basis of (7) and (1),

$$\begin{aligned} \sigma_{zx} &= p_x^- - \int_{-c}^{z} (\sigma_{xx,x}+\sigma_{yx,y})\,\mathrm{d}\zeta,\ldots, \\ \tau_{zx} &= q_x^- - \int_{-c}^{z} (\tau_{xx,x}+\tau_{yx,y}+\sigma_{zx}-\sigma_{xz})\,\mathrm{d}\zeta,\ldots, \end{aligned} \tag{8}$$

we have further, using equations (7) once more,

$$\begin{aligned} \int_{-c}^{c} (\sigma_{xx,x}+\sigma_{yx,y})\,\mathrm{d}\zeta+p_x^+-p_x^- &= 0,\ldots, \\ \int_{-c}^{c} (\tau_{xx,x}+\tau_{yx,y}+\sigma_{zx}-\sigma_{xz})\,\mathrm{d}\zeta+q_x^+-q_x^- &= 0,\ldots. \end{aligned} \tag{9}$$

The six equations (9) may be recognized, upon introduction of defining equations for stress resultants and couples, as the equilibrium equations of two-dimensional plate theory. At the same time, equations (3) and (5) are the associated two-dimensional compatibility equations. Equations (9), (5) and (3) as they stand, are specializations, in scalar form, of the corresponding vectorial equations for shells which are derived in [5].

For a two-dimensional theory of plates it is necessary to complement the system (9), (5) and (3) by a system of two-dimensional stress strain relations, that is, by a system of relations expressing stress resultants and couples in terms of the strain measures ε, γ, κ, λ. The derivation of such a two-dimensional system of relations, as a rational consequence of the three-dimensional system (2), (4), (6) and (8), may be said to represent the essence of the problem of deriving two-dimensional plate theory from three-dimensional elasticity theory.

3. THREE-DIMENSIONAL STRESS–STRAIN RELATIONS FOR A CLASS OF TRANSVERSELY ISOTROPIC PLATES

A system of stress–strain relations which is convenient in connection with the application of the proposed procedure and which at the same time leads to somewhat more general results than previously stated, is as follows:

$$\begin{aligned} &Ee_{xx} = \sigma_{xx}-\nu\sigma_{yy}-\nu_z\sigma_{zz}, \quad Ee_{xy} = (1+\nu)\sigma_{xy}, \quad Ge_{xz} = \sigma_{xz}, \\ &Ee_{yx} = (1+\nu)\sigma_{yx}, \quad Ee_{yy} = \sigma_{yy}-\nu\sigma_{xx}-\nu_z\sigma_{zz}, \quad Ge_{yz} = \sigma_{yz}, \\ &Ge_{zx} = \sigma_{zx}, \quad Ge_{zy} = \sigma_{zy}, \quad Ee_{zz} = (E/E_z)\sigma_{zz}-\nu_z\sigma_{xx}-\nu_z\sigma_{yy}, \end{aligned} \tag{10}$$

$$\begin{aligned} &\tau_{xx} = c^2\Gamma k_{xx}, \quad \tau_{xy} = c^2\Gamma k_{xy}, \quad \tau_{xz} = c^2\Lambda k_{xz}, \\ &\tau_{yx} = c^2\Gamma k_{yx}, \quad \tau_{yy} = c^2\Gamma k_{yy}, \quad \tau_{yz} = c^2\Lambda k_{yz}, \\ &\tau_{zx} = c^2\Gamma_z k_{zx}, \quad \tau_{zy} = c^2\Gamma_z k_{zy}, \quad \tau_{zz} = c^2\Lambda_z k_{zz}. \end{aligned} \tag{11}$$

In this the factor c^2 is introduced in order that the Γ and Λ have the same dimension as the E and G.

Vanishing of the coefficients Γ, Γ_z, Λ and Λ_z reduces the system (10) and (11) to a known form for media unable to support moment stresses, inasmuch as then $e_{xy} = e_{yx} = \frac{1}{2}\gamma_{xy}$, etc. in equations (10), as a consequence of $\sigma_{xy} = \sigma_{yx}$, etc.

4. INTEGRO-DIFFERENTIAL-EQUATIONS FORM OF THE PROBLEM

In order to derive a two-dimensional system of stress–strain relations from the three dimensional system (10) and (11) we first write (10) and (11) on the basis of equations (2), (4) and (8) as a system of integro-differential equations. It is sufficient to write out nine of these eighteen integro-differential equations. The remaining nine are analogous in form.

$$E\left\{\varepsilon_{xx}+z\kappa_{xx}+\int_0^z [e_{zx}-(\zeta-z)k_{zx}]_{,x}\,\mathrm{d}\zeta\right\}$$

$$= \sigma_{xx}-\nu\sigma_{yy}-\nu_z\left[p_z^- - \int_{-c}^z (\sigma_{xz,x}+\sigma_{yz,y})\,\mathrm{d}\zeta\right], \tag{12}$$

$$E\left\{\varepsilon_{xy}+z\kappa_{xy}-\int_0^z k_{zz}\,\mathrm{d}\zeta+\int_0^z [e_{zy}-(\zeta-z)k_{zy}]_{,x}\,\mathrm{d}\zeta\right\} = (1+\nu)\sigma_{xy}, \tag{13}$$

$$G\left\{\gamma_x+\int_0^z k_{zx}\,\mathrm{d}\zeta+\int_0^z e_{zz,x}\,\mathrm{d}\zeta\right\} = \sigma_{xz} \tag{14}$$

$$Ge_{zx} = p_x^- - \int_{-c}^z (\sigma_{xx,x}+\sigma_{yx,y})\,\mathrm{d}\zeta, \tag{15}$$

$$Ee_{zz} = (E/E_z)\left[p_z^- - \int_{-c}^z (\sigma_{xz,x}+\sigma_{yz,y})\,\mathrm{d}\zeta\right]-\nu_z(\sigma_{xx}+\sigma_{yy}), \tag{16}$$

$$c^2\Gamma\left\{\kappa_{xx}+\int_0^z k_{zx,x}\,\mathrm{d}\zeta\right\} = \tau_{xx}, \qquad c^2\Lambda\left\{\lambda_x+\int_0^z k_{zz,x}\,\mathrm{d}\zeta\right\} = \tau_{xz}, \tag{17, 18}$$

$$c^2\Gamma_z k_{zx} = q_x^- - \int_{-c}^z (\tau_{xx,x}+\tau_{yx,y}+\sigma_{zx}-\sigma_{xz})\,\mathrm{d}\zeta, \tag{19}$$

$$c^2\Lambda_z k_{zz} = q_z^- - \int_{-c}^z (\tau_{xz,x}+\tau_{yz,y}+\sigma_{xy}-\sigma_{yx})\,\mathrm{d}\zeta. \tag{20}$$

A two-dimensional theory as a rational consequence of the system (12) to (20) is now based on the concept of a smallest characteristic length L, this length being large compared to the plate thickness $2c$. Briefly, L is that length which permits writing the order of magnitude relations:

$$e_{zx,x} = O(e_{zx}/L), \qquad \sigma_{xz,z} = O(\sigma_{xz}/L), \text{ etc.} \tag{21}$$

Equations (21) may be used systematically by introducing dimensionless coordinates $\hat{x} = Lx$, $\hat{y} = Ly$, $\hat{z} = cz$ and by expanding the solutions of the resulting system of equations, having independent variables $\hat{x}$, $\hat{y}$, $\hat{z}$, asymptotically in powers of the small parameter c/L. For the present purposes it is more convenient, and effectively equivalent, to make use of Goldenweiser's concept of iterative procedures in connection with the step from three to two dimensions [1].

As a point of departure for the iterative procedure, we rewrite part of the system (12) to (20) with the help of equations (1), (5) and (8) in a form more convenient for our purposes, as follows:

$$E\left\{\varepsilon_{xx}+z\kappa_{xx}+\int_0^z [e_{zx}-(\zeta-z)k_{zx}]_{,x}\,\mathrm{d}\zeta\right. = \sigma_{xx}-\nu\sigma_{yy}$$

$$-\nu_z\left\{p_z^- - \int_{-c}^{z}\left[p_z^- - \int_{-c}^{\zeta}(\sigma_{xx,x}+\sigma_{yx,y})\,\mathrm{d}\eta+\tau_{xx,x}+\tau_{yx,y}+\tau_{zx,\zeta}\right]_{,x}\mathrm{d}\zeta\right.$$

$$\left.-\int_{-c}^{z}\left[p_y^- - \int_{-c}^{\zeta}(\sigma_{xy,x}+\sigma_{yy,y})\,\mathrm{d}\eta+\tau_{xy,x}+\tau_{yy,y}+\tau_{zy,\zeta}\right]_{,y}\mathrm{d}\zeta\right\}, \qquad (12^*)$$

$$E\left\{\varepsilon_{xy}+\tfrac{1}{2}z(\kappa_{xy}+\kappa_{yx})+\tfrac{1}{2}z(\gamma_{y,x}-\gamma_{x,y})-\int_0^z k_{zz}\,\mathrm{d}\zeta+\int_0^z [e_{zy}-(\zeta-z)k_{zy}]_{,x}\,\mathrm{d}\zeta\right\}$$

$$=\tfrac{1}{2}(1+\nu)(\sigma_{xy}+\sigma_{yx})-\tfrac{1}{2}(1+\nu)(\tau_{xz,x}+\tau_{yz,y}+\tau_{zz,z}), \qquad (13^*)$$

$$G\left\{\gamma_x+\int_0^z k_{zx}\,\mathrm{d}\zeta+\int_0^z e_{zz,x}\,\mathrm{d}\zeta\right\} = p_x^- - \int_{-c}^{z}(\sigma_{xx,x}+\sigma_{yx,y})\,\mathrm{d}\zeta+\tau_{xx,x}+\tau_{yx,y}+\tau_{zx,z}, \quad (14^*)$$

$$Ee_{zz} = (E/E_z)\left\{p_z^- - \int_{-c}^{z}\left[p_x^- - \int_{-c}^{\zeta}(\sigma_{xx,x}+\sigma_{yx,y})\,\mathrm{d}\eta+\tau_{xx,x}+\tau_{yx,y}+\tau_{zx,\zeta}\right]_{,x}\mathrm{d}\zeta\right.$$

$$\left.-\int_{-c}^{z}\left[p_y^- - \int_{-c}^{\zeta}(\sigma_{xy,x}+\sigma_{yy,y})\,\mathrm{d}\eta+\tau_{xy,x}+\tau_{yy,y}+\tau_{zy,\zeta}\right]_{,y}\mathrm{d}\zeta\right\}-\nu_z(\sigma_{xx}+\sigma_{yy}), \quad (16^*)$$

$$c^2\Lambda\left(\varepsilon_{yx,x}-\varepsilon_{xx,y}+\int_0^z k_{zz,x}\,\mathrm{d}\zeta\right) = \tau_{xz}, \qquad (18^*)$$

$$c^2\Gamma_z k_{zx} = \tau_{zx}, \qquad c^2\Lambda_z k_{zz} = \tau_{zz}. \qquad (19^*, 20^*)$$

The corresponding equations (15*) and (17*) are the same as (15) and (17), respectively.

Equations (12*) to (20*) involve the stresses σ_{xz}, σ_{yz}, σ_{zz}, σ_{zx}, σ_{zy} implicitly only. In order to take account of the boundary conditions for $z = \pm c$ for σ_{zx}, σ_{zy} and σ_{zz} the system (12*) to (20*) is complemented by the relations

$$p_x^+ - p_x^- + \int_{-c}^{c}(\sigma_{xx,x}+\sigma_{yx,y})\,\mathrm{d}\zeta = 0, \qquad p_y^+ - p_y^- + \int_{-c}^{c}(\sigma_{xy,x}+\sigma_{yy,y})\,\mathrm{d}\zeta = 0, \qquad (22)$$

$$p_z^+ - p_z^- + \int_{-c}^{c}\left[p_x^- - \int_{-c}^{\zeta}(\sigma_{xx,x}+\sigma_{yx,y})\,\mathrm{d}\eta+\tau_{xx,x}+\tau_{yx,y}\right]_{,x}\mathrm{d}\zeta$$

$$+\int_{-c}^{c}\left[p_y^- - \int_{-c}^{\zeta}(\sigma_{xy,x}+\sigma_{yy,y})\,\mathrm{d}\eta+\tau_{xy,x}+\tau_{yy,y}\right]_{,y}\mathrm{d}\zeta = 0. \qquad (23)$$

At the same time, equations (12*) to (20*) remain subject to the six boundary conditions (7) for τ_{zx}, τ_{zy} and τ_{zz}.

For simplicity's sake it will be assumed from here on that the only tractions applied to the faces $z = \pm c$ are those corresponding to the case of transverse bending that is we will set

$$p_x^\pm = p_y^\pm = q_x^\pm = q_y^\pm = q_z^\pm = 0, \qquad p_z^\pm = \pm\tfrac{1}{2}p. \qquad (24)$$

5. THE ITERATIVE PROCEDURE

We use as *a priori* assumptions for an iterative procedure that x and y-derivative terms in the integro-differential equations (12*) to (20*) will be small compared to all other terms. Designating quantities pertaining to the nth iterative step by $\varepsilon_{xx}^{(n)}$, etc. this means that we solve (12*) to (20*) (with the assumed simplifications in regard to face tractions) by writing for $n = 1, 2, 3, \ldots$

$$E\left\{\varepsilon_{xx}^{(n)}+z\kappa_{xx}^{(n)}+\int_0^z [e_{zx}^{(n-1)}-(\zeta-z)k_{zx}^{(n-1)}]_{,x}\,\mathrm{d}\zeta = \sigma_{xx}^{(n)}-\nu\sigma_{yy}^{(n)}\right.$$
$$-\nu_z\left\{\tfrac{1}{2}p+\int_{-c}^{z}\left[\int_{-c}^{\zeta}(\sigma_{xx,x}^{(n-2)}+\sigma_{yx,y}^{(n-2)})\,\mathrm{d}\eta-\tau_{xx,x}^{(n-2)}-\tau_{yx,y}^{(n-2)}-\tau_{zx,\zeta}^{(n-1)}\right]_{,x}\mathrm{d}\zeta\right.$$
$$\left.+\int_{-c}^{z}\left[\int_{-c}^{\zeta}(\sigma_{xy,x}^{(n-2)}+\sigma_{yy,y}^{(n-2)})\,\mathrm{d}\eta-\tau_{xy,x}^{(n-2)}-\tau_{yy,y}^{(n-2)}-\tau_{zy,\zeta}^{(n-1)}\right]_{,y}\mathrm{d}\zeta\right\}, \tag{25}$$

$$E\left\{\varepsilon_{xy}^{(n)}+\tfrac{1}{2}z(\kappa_{xy}^{(n)}+\kappa_{yx}^{(n)})+\tfrac{1}{2}z(\gamma_{y,x}^{(n-1)}-\gamma_{x,y}^{(n-1)})-\int_0^{\zeta}k_{zz}^{(n)}\,\mathrm{d}\zeta+\int_0^z[e_{zy}^{(n-1)}-(\zeta-z)k_{xy}^{(n-1)}]_{,x}\,\mathrm{d}\zeta\right\}$$
$$=\tfrac{1}{2}(1+\nu)(\sigma_{xy}^{(n)}+\sigma_{yx}^{(n)})-\tfrac{1}{2}(1+\nu)(\tau_{xz,x}^{(n-1)}+\tau_{yz,y}^{(n-1)}+\tau_{zz,z}^{(n)}), \tag{26}$$

$$G\left[\gamma_x^{(n)}+\int_0^z k_{zx}^{(n)}\,\mathrm{d}\zeta+\int_0^z e_{zz,x}^{(n-1)}\,\mathrm{d}\zeta\right]$$
$$=-\int_c^z(\sigma_{xx,x}^{(n-1)}+\sigma_{yx,y}^{(n-1)})\,\mathrm{d}\zeta+\tau_{xx,x}^{(n-1)}+\tau_{yx,y}^{(n-1)}+\tau_{zx,z}^{(n)}, \tag{27}$$

$$Ge_{zx}^{(n)} = -\int_{-c}^{z}(\sigma_{xx,x}^{(n-1)}+\sigma_{yx,y}^{(n-1)})\,\mathrm{d}\zeta, \tag{28}$$

$$Ee_{zz}^{(n)} = (E/E_z)\left\{-\tfrac{1}{2}p+\int_{-c}^{z}\left[\int_{-c}^{\zeta}(\sigma_{xx,x}^{(n-2)}+\sigma_{yx}^{(n-2)})\,\mathrm{d}\eta+\tau_{xx,x}^{(n-2)}+\tau_{yx,y}^{(n-2)}+\tau_{zx,\zeta}^{(n-1)}\right]_{,x}\mathrm{d}\zeta\right.$$
$$\left.+\int_{-c}^{z}\left[\int_{-c}^{\zeta}(\sigma_{xy,x}^{(n-2)}+\sigma_{yy,y}^{(n-2)})\,\mathrm{d}\eta+\tau_{xy,x}^{(n-2)}+\tau_{yy,y}^{(n-2)}+\tau_{zy,\zeta}^{(n-1)}\right]_{,y}\mathrm{d}\zeta\right\}-\nu_z(\sigma_{xx}^{(n)}+\sigma_{yy}^{(n)}), \tag{29}$$

$$c^2\Gamma\left\{\kappa_{xx}^{(n)}+\int_0^z k_{zx,x}^{(n-1)}\,\mathrm{d}\zeta\right\} = \tau_{xx}^{(n)}, \tag{30}$$

$$c^2\Lambda\left\{\varepsilon_{yx,x}^{(n-1)}-\varepsilon_{xx,y}^{(n-1)}+\int_0^z k_{zz,x}^{(n-1)}\,\mathrm{d}\zeta\right\} = \tau_{xz}^{(n)} \tag{31}$$

$$c^2\Gamma_z k_{zx}^{(n)} = \tau_{zx}^{(n)}, \qquad c^2\Lambda_z k_{zz}^{(n)} = \tau_{zz}^{(n)}, \tag{32, 33}$$

with $e_{zx}^{(0)}$, $\sigma_{xx}^{(-1)}$, etc. all being zero, by definition, and with p being of the same order as the terms associated with it.

In addition to this, equations (22) and (23) imply the relations

$$\int_{-c}^{c} (\sigma_{xx,x}^{(n)} + \sigma_{yx,y}^{(n)})\,d\zeta = 0, \qquad \int_{-c}^{c} (\sigma_{xy,x}^{(n)} + \sigma_{yy,y}^{(n)})\,d\zeta = 0, \tag{34}$$

$$\begin{aligned} p - \int_{-c}^{c} \left[\int_{-c}^{\zeta} (\sigma_{xx,x}^{(n)} + \sigma_{yx,y}^{(n)})\,d\eta + \tau_{xx,x}^{(n)} + \tau_{yx,y}^{(n)} \right]_{,x} d\zeta \\ - \int_{-c}^{c} \left[\int_{-c}^{\zeta} (\sigma_{xy,x}^{(n)} + \sigma_{yy,y}^{(n)})\,d\eta + \tau_{xy,x}^{(n)} + \tau_{yy,y}^{(n)} \right]_{,y} d\zeta = 0, \end{aligned} \tag{35}$$

and the face boundary conditions for τ_{zx}, τ_{zy}, τ_{zz} are written in the form

$$z = \pm c, \qquad \tau_{zx}^{(n)} = 0, \qquad \tau_{zy}^{(n)} = 0, \qquad \tau_{zz}^{(n)} = 0. \tag{36}$$

Finally, the associated two-dimensional compatibility equations which follow from (3) and (5) are written as

$$\varepsilon_{yy,xx}^{(n)} - (\varepsilon_{yx}^{(n)} + \varepsilon_{xy}^{(n)})_{,xy} + \varepsilon_{xx,yy}^{(n)} = 0, \tag{37}$$

$$\tfrac{1}{2}(\kappa_{yx}^{(n)} + \kappa_{xy}^{(n)})_{,x} - \kappa_{xx,y}^{(n)} = \tfrac{1}{2}(\gamma_{x,y}^{(n-1)} - \gamma_{y,x}^{(n-1)})_{,x}, \tag{38}$$

$$\kappa_{yy,x}^{(n)} - \tfrac{1}{2}(\kappa_{xy}^{(n)} + \kappa_{yx}^{(n)})_{,y} = \tfrac{1}{2}(\gamma_{y,x}^{(n-1)} - \gamma_{x,y}^{(n-1)})_{,y}. \tag{39}$$

6. THE EQUATIONS OF THE FIRST STEP OF THE ITERATIVE PROCEDURE

Equations (25) to (39), for the case $n = 1$, reduce to

$$E\{\varepsilon_{xx}^{(1)} + z\kappa_{xx}^{(1)}\} = \sigma_{xx}^{(1)} - \nu\sigma_{yy}^{(1)}, \tag{40}$$

$$E\left\{\varepsilon_{xy}^{(1)} + \tfrac{1}{2}z(\kappa_{xy}^{(1)} + \kappa_{yx}^{(1)}) - \int_0^z k_{zz}^{(1)}\,d\zeta\right\} = \tfrac{1}{2}(1+\nu)(\sigma_{xy}^{(1)} + \sigma_{yx}^{(1)} - \tau_{zz,z}^{(1)}), \tag{41}$$

$$G\left\{\gamma_x^{(1)} + \int_0^z k_{zx}^{(1)}\,d\zeta\right\} = \tau_{zx,x}^{(1)}, \tag{42}$$

$$Ge_{zx}^{(1)} = 0, \qquad Ee_{zz}^{(1)} = -\nu_z(\sigma_{xx}^{(1)} + \sigma_{yy}^{(1)}), \tag{43}$$

$$c^2\Gamma\kappa_{xx}^{(1)} = \tau_{xx}^{(1)}, \qquad 0 = \tau_{xz}^{(1)}, \tag{44}$$

$$c^2\Gamma_z k_{zx}^{(1)} = \tau_{zx}^{(1)}, \qquad c^2\Lambda_z k_{zz}^{(1)} = \tau_{zz}^{(1)}. \tag{45}$$

Equations (34) to (37) remain as is with "(1)" written for "(n)" and equations (38) and (39) become

$$\tfrac{1}{2}(\kappa_{yx}^{(1)} + \kappa_{xy}^{(1)})_{,x} - \kappa_{xx,y}^{(1)} = 0, \qquad \kappa_{yy,x}^{(1)} - \tfrac{1}{2}(\kappa_{xy}^{(1)} + \kappa_{yx}^{(1)})_{,y} = 0. \tag{46}$$

It follows next from (41) and the corresponding equation involving $\varepsilon_{yx}^{(1)}$ that

$$E\left\{\varepsilon_{xy}^{(1)} - \varepsilon_{yx}^{(1)} - 2\int_0^z k_{zz}^{(1)}\,d\zeta\right\} = (1+\nu)\tau_{zz,z}^{(1)}. \tag{47}$$

This, together with the second equation in (45) and with the boundary conditions (36) for $\tau_{zz}^{(1)}$, gives

$$\varepsilon_{xy}^{(1)} = \varepsilon_{yx}^{(1)}, \qquad k_{zz}^{(1)} = 0, \qquad \tau_{zz}^{(1)} = 0. \tag{48}$$

Similarly, it follows from (42), the first equation in (45) and the boundary conditions (36) for $\tau_{zx}^{(1)}$ that

$$\gamma_x^{(1)} = 0, \qquad k_{zx}^{(1)} = 0, \qquad \tau_{zx}^{(1)} = 0. \tag{49}$$

Having (48) and (49) we have then as expressions for the non-vanishing components of stress implied by the equations of the first step of the iterative procedure

$$(1-\nu^2)\sigma_{xx}^{(1)} = E(\varepsilon_{xx}^{(1)} + \nu\varepsilon_{yy}^{(1)} + z\kappa_{xx}^{(1)} + \nu z\kappa_{yy}^{(1)}), \qquad (1-\nu^2)\sigma_{yy}^{(1)} = \ldots, \tag{50}$$

$$(1+\nu)\sigma_{xy}^{(1)} = (1+\nu)\sigma_{yx}^{(1)} = \tfrac{1}{2}(\varepsilon_{xy}^{(1)} + \varepsilon_{yx}^{(1)} + z\kappa_{xy}^{(1)} + z\kappa_{yx}^{(1)}), \tag{51}$$

$$\tau_{xx}^{(1)} = c^2\Gamma\kappa_{xx}^{(1)}, \qquad \tau_{xy}^{(1)} = \tfrac{1}{2}c^2\Gamma(\kappa_{xy}^{(1)} + \kappa_{yx}^{(1)}), \ldots. \tag{52}$$

The six functions $\varepsilon_{xx}^{(1)}$, $\varepsilon_{xy}^{(1)} + \varepsilon_{yx}^{(1)}$, $\kappa_{xx}^{(1)}$, $\kappa_{yy}^{(1)}$, $\kappa_{xy}^{(1)} + \kappa_{yx}^{(1)}$ in this are subject to three compatibility equations consisting of the two equations (46) and of

$$\varepsilon_{yy,xx}^{(1)} - (\varepsilon_{xy}^{(1)} + \varepsilon_{yx}^{(1)})_{,xy} + \varepsilon_{xx,yy}^{(1)} = 0, \tag{53}$$

and of three equilibrium equations, which in accordance with (34) and (35) are of the form,

$$\int_{-c}^{c} (\sigma_{xx,x}^{(1)} + \sigma_{yx,y}^{(1)})\,\mathrm{d}\zeta = 0, \qquad \int_{-c}^{c} (\sigma_{xy,x}^{(1)} + \sigma_{yy,y}^{(1)})\,\mathrm{d}\zeta = 0, \tag{54}$$

$$\int_{-c}^{c} \left\{ \int_{-c}^{\zeta} [(\sigma_{xx,x}^{(1)} + \sigma_{yx,y}^{(1)})_{,x} + (\sigma_{xy,x}^{(1)} + \sigma_{yy,y}^{(1)})_{,y}]\,\mathrm{d}\eta \right. \\ \left. + (\tau_{xx,x}^{(1)} + \tau_{yx,y}^{(1)})_{,x} + (\tau_{xy,x}^{(1)} + \tau_{yy,y}^{(1)})_{,y} \right\} \mathrm{d}\zeta = p, \tag{55}$$

Equations (50) to (55) may be recognized as the results of classical thin-plate theory, generalized by the incorporation of the moment stress terms $\tau^{(1)}$, in the form of six differential equations for the six quantities $\varepsilon_{xx}^{(1)}$, $\varepsilon_{yy}^{(1)}$, $\varepsilon_{xy}^{(1)} + \varepsilon_{yx}^{(1)}$, $\kappa_{xx}^{(1)}$, $\kappa_{yy}^{(1)}$ and $\kappa_{xy}^{(1)} + \kappa_{yx}^{(1)}$. The order of this system is not changed by the incorporation of the moment stress terms and, accordingly, an appropriate version of the classical Kirchhoff boundary conditions is associated with these equations.

The remaining measures of strain and stress, as given by the first step of the procedure are $e_{zx}^{(1)}$ (and $e_{zy}^{(1)}$) and $e_{zz}^{(1)}$ as in (43), while on the basis of (8) and (1) and consistent with the foregoing,

$$\sigma_{zx}^{(1)} = -\int_{-c}^{z} (\sigma_{xx,x}^{(1)} + \sigma_{yx,y}^{(1)})\,\mathrm{d}\zeta, \tag{56}$$

and

$$\sigma_{xz}^{(1)} = \sigma_{zx}^{(1)} + \tau_{xx,x}^{(1)} + \tau_{yx,y}^{(1)}, \tag{57}$$

with corresponding expressions for $\sigma_{zy}^{(1)}$ and $\sigma_{yz}^{(1)}$, and

$$\sigma_{zz}^{(1)} = -\tfrac{1}{2}p - \int_{-c}^{z} (\sigma_{xz,x}^{(1)} + \sigma_{yz,y}^{(1)})\,\mathrm{d}\zeta. \tag{58}$$

7. RANGE OF VALIDITY OF ITERATIVE PROCEDURE

In speaking of *an* iterative procedure we admit the possibility of procedures differing from the one proposed in Section 5. To see the limitations of the proposed procedure we note that the characteristic length L associated with the "classical" system of differential equations (46) and (53) to (55) will be the smallest geometrically representative width of the plate. It is then seen that the perturbation terms in the second step of the procedure will be of order c/L and of order c^2/L^2—except for the effect of differences in orders of magnitude in the elasticity coefficients—relative to the quantities $\varepsilon_{xx}^{(2)}$, etc. which are to be determined. Assuming v and v_z of order unity we will then deduce relations such as:

$$\frac{c^2}{L^2}\frac{E}{G} \ll 1, \qquad \frac{c^2}{L^2}\frac{\Gamma}{G} \ll 1, \qquad \frac{c\Lambda}{LE} \ll 1,$$

as conditions for the validity of the iterative procedure as proposed.

Physically, the principal meaning of these restrictions is that when E and Γ are bounded from above in this manner, (or G bounded from below) then transverse shear deformation comes out as a higher-order-of-smallness effect, leading automatically to a Kirchhoff-type theory. When E and/or Γ are not bounded as in (59) then transverse shear deformation will play a role and not all x and y-derivative terms in the integro-differential equations (12*) to (20*) may be relegated to the class of higher-order-of smallness terms, as is done in the system (25) to (33). It is in this sense that the asymptotic theory of Green and Naghdi [2] appears to be one for which $c^2\Gamma/L^2G = O(1)$ while at the same time $E \ll \Gamma$.

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Абстракт—При разработке недавно предлагаемого метода решения двухмерной теории оболочек на основе трехмерной теории упругости, исследуется подробно специальный случай плоской пластинки. В аспекте начальной точки, принимается соответствующий вариант теории упругости, заключающий как моментные напряжения, так и напряжения вызывающие силами. Указывается, что некоторая прямая, но не раньше учитываемая редукция трехмерных уравнений равновесия и сплошности приводит к соответствующим двухмерным чреьмемцлм равновесия так и сплошности, несмотря на сохранение трехмерных аспектов задачи, в форме системы интегро-дифференциальных определяющих уравнений. Решение этих двухмерных уравнений оказывается тогда единственным методом, который проявляет параметрическое разложение или итерацию, в связи с условием наименьшей характеристической длины при сравнении к толщине пластинки.